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APPROXIMATE THEORIES FOR WAVE PROPAGATION AND VIBRATIONS IN ELASTIC RINGS AND HELICAL COILS OF SMALL PITCH

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Abstract—From the equations of linear elasticity, three levels of approximate theories are derived for in-plane deformation and motion of thin, circular rings. The accuracy of each theory is determined by comparison with harmonic wave solutions of the elasticity solution. Boundary conditions for uniqueness are established. The results may also be applied to helical coils of small pitch and to cylindrical shells when the equations are converted to plane strain.

INTRODUCTION

Many authors have derived one-dimensional equations capable of describing in-plane vibrations and wave propagation in elastic, circular rings. These *ad-hoc* approaches have led to a wide variety of approximate theories which are generalizations of the straight bar equations, i.e. Bernoulli–Euler and Timoshenko beam equations, and classical extensional bar equations. In the case of the ring, extensional motion is coupled to flexure and shear, thus complicating the analysis. Each of these approximate theories has a limitation on the range of frequencies for which it is reasonably accurate. Until the availability of the recently developed exact solution[1] of the equations of generalized plane stress for the case of waves in a circular ring, it was impossible to establish this frequency range without extensive experimentation. Such experimentation as was performed was quite limited and overlooked were the imaginary and complex segments of the dispersion curves and the behavior at low frequency and very long wave length.

In this paper we establish the appropriate frequency ranges for each of the four significant theories, and, for some of the theories, determine that their equations contain extraneous terms. Employing the procedure introduced by Mindlin for plates[2], we start with the variational equations of elasticity, employ power series expansions of the radial coordinate, utilize stress–strain relations for generalized plane stress, and truncate the series suitably to obtain our three-mode theory for rings, in which flexure, shear, and extension are coupled. This theory has three degrees of freedom—radial and tangential displacements, and rotation of the cross-section. Though he did not solve them, Bresse obtained the equivalent equations in 1859[3]. With the proper choice of shear correction factor, it is shown that these equations are valid in the same wide frequency range as the corrected Timoshenko beam equations for straight bars. Despite numerous more recent efforts, Bresse's theory remains as the most general, and contains no extraneous terms. In our systematic derivation, it must be assumed that the ring is thin as well as narrow compared to the ring diameter. Therefore, this theory and the three subsequently discussed are limited to thin rings.

By constraining the shear deformation, leaving only two degrees of freedom, and neglecting rotatory inertia, we reduce the three-mode theory to the simpler two-mode theory which is identical with that first derived by Waltking in 1934[5]. It is shown that the simpler two-mode equations, coupling flexure and extension are valid in the same frequency range as the Bernoulli–Euler beam equations, i.e. less than 10 per cent of the thickness shear cut-off frequency.

Two types of one-mode theories can be obtained with further constraints. The first is the flexure theory which is derived from the two-mode equations by constraining the extensional deformation, leaving only one degree of freedom. Though he did not present the equations of motion and deformation in an easily understood form, Hoppe derived the correct frequency equation for this case in 1871[6]. We show that these equations may be applied for a range of frequencies nearly as great as the two-mode theory in most cases.

The other one-mode theory is a special case and must be handled with care. By setting the bending stiffness equal to zero and further constraining the displacements in the two-mode theory, a one-mode extensional theory is obtained which gives very good results for *one branch* in the band of frequencies between the ring mode and the limit of classical extensional theory in straight bars. Because there are propagating flexual waves at these frequencies, solutions of the equations of this theory must be carefully interpreted. Hoppe was again the first to present the appropriate frequency equation for this case[6].

All of these theories may be applied to helical coils of small pitch. With the simple replacement of Young's modulus E by $E/(1 - \nu^2)$ where ν is Poisson's ratio, each theory can be applied to cylindrical shells where plane strain rather than generalized plane stress prevails. Boundary conditions for finite ring segments are presented for each of the four cases.

BASIC EQUATIONS

Consider the elastic ring segment shown in Fig. 1. The ring has rectangular cross-section with traction-free conditions at $r = a$, $r = b$, and the two parallel sides perpendicular to the axis of the ring. The ring thickness $h(= b - a)$ is small compared to the mean diameter of the ring, i.e. $h \ll a + b$. If only in-plane motion of the ring is to be considered, the equations of generalized plane stress are appropriate and the independent variables are r, θ and time t. From application of Hamilton's principle for an elastic medium without body forces[7],

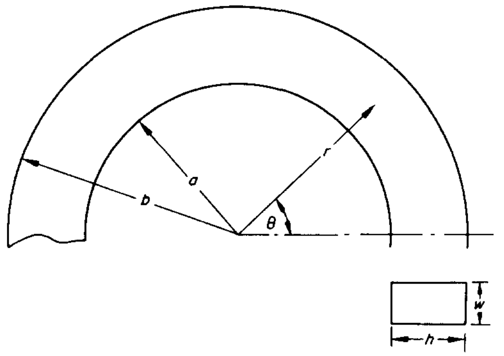

Fig. 1. Circular ring segment, average radius $R = (a + b)/2$.

two variational equations are obtained:

$$
\begin{aligned}
&\int_{\theta_1}^{\theta_2}\int_{R-h/2}^{R+h/2}\left[\tau_{rr,r}+\frac{\tau_{r\theta,\theta}}{r}+\frac{(\tau_{rr}-\tau_{\theta\theta})}{r}-\rho u_{r,tt}\right]\delta u_r r\,\mathrm{d}r\,\mathrm{d}\theta=0,\\
&\int_{\theta_1}^{\theta_2}\int_{R-h/2}^{R+h/2}\left[\tau_{r\theta,r}+\frac{\tau_{\theta\theta,\theta}}{r}+\frac{2\tau_{r\theta}}{r}-\rho u_{\theta,tt}\right]\delta u_\theta r\,\mathrm{d}r\,\mathrm{d}\theta=0,
\end{aligned}
\tag{1}
$$

where the comma denotes partial differentiation with respect to the indices that follow, ρ is the density, τ_{rr}, $\tau_{\theta\theta}$, $\tau_{r\theta}$ are the stress components and u_r, u_θ are the displacement components in polar coordinates. The stress–strain relations of generalized plane stress are

$$
\begin{aligned}
\tau_{rr}&=[E/(1-v^2)](\varepsilon_{rr}+v\varepsilon_{\theta\theta}),\\
\tau_{\theta\theta}&=[E/(1-v^2)](\varepsilon_{\theta\theta}+v\varepsilon_{rr}),\\
\tau_{r\theta}&=2G\varepsilon_{r\theta},
\end{aligned}
\tag{2}
$$

where E is Young's modulus, v is Poisson's ratio, and $G(=E/2(1+v))$ is the shear modulus. The strains are related to the displacements by

$$
\begin{aligned}
\varepsilon_{rr}&=u_{r,r},\\
\varepsilon_{\theta\theta}&=u_r/r+u_{\theta,\theta}/r,\\
\varepsilon_{r\theta}&=[u_{\theta,r}+(u_{r,\theta}-u_\theta)/r]/2.
\end{aligned}
$$

In order to obtain one-dimensional equations, we take power series expansions of the displacement functions about the mid-line defined by $R=(a+b)/2$:

$$
\begin{aligned}
u_r&=\sum_{n=0}^{\infty}x^n u_r^{(n)},\\
u_\theta&=\sum_{n=0}^{\infty}x^n u_\theta^{(n)},
\end{aligned}
\tag{3}
$$

in which $u_r^{(n)}$ and $u_\theta^{(n)}$ are functions of θ and t only (not r) and $x=r-R$ is the radial distance from the mid-line. The displacement variations in (1) become

$$
\begin{aligned}
\delta u_r&=\sum_{n=0}^{\infty}x^n\delta u_r^{(n)},\\
\delta u_\theta&=\sum_{n=0}^{\infty}x^n\delta u_\theta^{(n)}.
\end{aligned}
\tag{4}
$$

In the same way the strains may be expanded:

$$
\begin{aligned}
\varepsilon_{rr}&=\sum_{n=0}^{\infty}x^n\varepsilon_{rr}{}^{(n)},\\
\varepsilon_{\theta\theta}&=\sum_{n=0}^{\infty}x^n\varepsilon_{\theta\theta}{}^{(n)},\\
\varepsilon_{r\theta}&=\sum_{n=0}^{\infty}x^n\varepsilon_{r\theta}{}^{(n)}.
\end{aligned}
$$

We can define the nth order stresses by

$$\tau_{rr}^{(n)} = \int_{-h/2}^{h/2} x^n \tau_{rr}\, \mathrm{d}x,$$
$$\tau_{\theta\theta}^{(n)} = \int_{-h/2}^{h/2} x^n \tau_{\theta\theta}\, \mathrm{d}x, \tag{5}$$
$$\tau_{r\theta}^{(n)} = \int_{-h/2}^{h/2} x^n \tau_{r\theta}\, \mathrm{d}x.$$

We are interested in developing approximate equations capable of accurately describing harmonic waves propagating with frequencies as high as those covered by the Timoshenko beam equations for a straight bar of similar cross-section. The generalized plane stress assumption precludes the description of frequencies higher than this range[1]. We follow the course suggested by Mindlin's development of approximate plate equations[2]. All series are truncated above the terms corresponding to $n = 1$. Then we provide for the free development of the strain ε_{rr} by setting $\tau_{rr} = 0$, and obtaining from (2a), $\varepsilon_{rr} = -\nu\varepsilon_{\theta\theta}$ and from (5a), $\tau_{rr}^{(0)} = \tau_{rr}^{(1)} = 0$. Next we drop inertia terms involving $u^1_{r,tt}$. In addition to these approximations which follow directly from the plate studies, we impose the requirement of relatively thin rings: $h/2 \ll R$.

Under these restrictions, we substitute (3) and (4) into (1), carry out the integrations on r, apply the definitions (5) and impose the requirement of independent displacement variations over the arbitrary interval $\theta_1 - \theta_2$ and obtain three stress equations of motion:

$$\tau^{(0)}_{r\theta,\theta} - \tau^{(0)}_{\theta\theta} = R\rho h u^{(0)}_{r,tt},$$
$$\tau^{(0)}_{\theta\theta,\theta} + \tau^{(0)}_{r\theta} = R\rho h u^{(0)}_{\theta,tt}, \tag{6}$$
$$\tau^{(1)}_{\theta\theta,\theta} - R\tau^{(0)}_{r\theta} = R\rho (h^3/R) u^{(1)}_{\theta,tt}.$$

The stress–strain relations become

$$\tau^{(0)}_{\theta\theta} = Eh\varepsilon^{(0)}_{\theta\theta},$$
$$\tau^{(1)}_{\theta\theta} = E(h^3/R)\varepsilon^{(1)}_{\theta\theta}, \tag{7}$$
$$\tau^{(0)}_{r\theta} = \kappa^2 Gh\varepsilon^{(0)}_{r\theta},$$

and the strain–displacement relations are

$$\varepsilon^{(0)}_{\theta\theta} = (u^{(0)}_r + u^{(0)}_{\theta,\theta})/R,$$
$$\varepsilon^{(1)}_{\theta\theta} = (1/R)u^{(1)}_{\theta,\theta}, \tag{8}$$
$$\varepsilon^{(0)}_{r\theta} = (Ru^{(1)}_\theta + u^{(0)}_{r,\theta} - u^{(0)}_\theta)/2R.$$

The factor κ^2 in (7c) does not appear as a consequence of the systematic development described but is inserted in order to match the thickness–shear frequency with that obtained from the generalized plane stress solution[1] hereafter referred to as the "exact solution". With this simple adjustment, the dispersion curves are highly accurate throughout the same wide frequency range as the similarly corrected Timoshenko beam theory[4].

It is useful to adopt a different notation and rewrite (6)–(8) in a form which is perhaps more recognizable. This is done in the next section.

THREE-MODE THEORY

We introduce the variable $S = R\theta$, which is the distance measured along the mid-line. Then we rewrite (6) as

$$\begin{aligned} \frac{\partial Q}{\partial S} - \frac{P}{R} &= \rho A \frac{\partial^2 u}{\partial t^2}, \\ \frac{\partial P}{\partial S} + \frac{Q}{R} &= \rho A \frac{\partial^2 v}{\partial t^2}, \\ \frac{\partial M}{\partial S} - Q &= \rho I \frac{\partial^2 \psi}{\partial t^2}, \end{aligned} \tag{9}$$

and after combining (7) and (8), we rewrite them as

$$\begin{aligned} P &= EA(u/R + \partial v/\partial S), \\ M &= EI(\partial \psi/\partial S), \\ Q &= \kappa^2 GA(\psi + \partial u/\partial S - v/R), \end{aligned} \tag{10}$$

where

$A = wh$, the cross-sectional area,
$I = wh^3/12$, the second moment of area with respect to the axis through and perpendicular to the mid-line,
$u = u_r^{(0)}$, the radial displacement of the cross-section (See Fig. 2),
$v = u_\theta^{(0)}$, the tangential displacement of the cross-section (See Fig. 2),
$\psi = u_\theta^{(1)}$, the rotation of the cross-section (See Fig. 2),

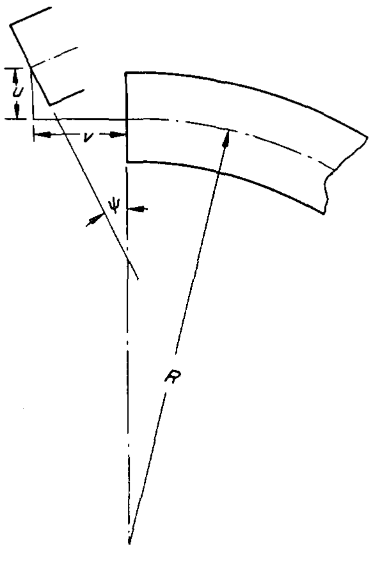

Fig. 2. Displacement components of ring cross-section: radial displacement u, tangential displacement v, rotation ψ.

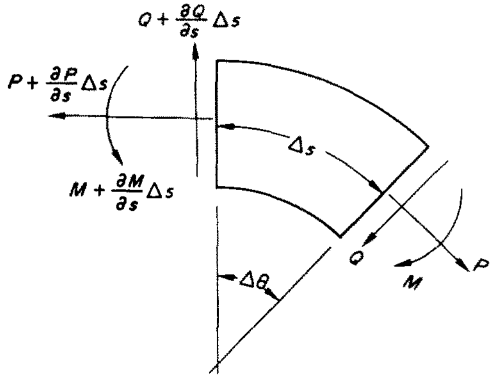

Fig. 3. Resultant forces and moment on ring cross-section.

$P = w\tau_{\theta\theta}^{(0)}$, the resultant tangential force (See Fig. 3),
$Q = w\tau_{r\theta}^{(0)}$, the resultant radial shear force (See Fig. 3),
$M = w\tau_{\theta\theta}^{(1)}$, the resultant bending moment (See Fig. 3).

By a theorem analogous to Neumann's[8], it can be shown that for a bounded ring segment, a unique solution of (9) and (10) is obtained when at each end one member of each of the following three products is specified:

$$Pv,\ Qu,\ M\psi. \tag{11}$$

We will refer to (9) and (10) hereafter as the "three-mode" equations since for a given wave length, there are three possible frequencies of propagating harmonic waves. Inspection of the equations reveal coupled flexural, shear, and extensional deformation. Radial, tangential, and rotatory inertia are represented.

For rings of rectangular cross-section the shear correction factor κ^2 can be taken as $\pi^2/12$, just as in straight bars and plates. Appropriate values of κ^2 for other cross-sectional shapes are given by Mindlin and Deresiewicz[4].

It appears to have been overlooked that Bresse published equations equivalent to (9) and (10) in 1859[3]. Morley[9] and Graff[10] have recently published less elegant equations containing extraneous terms when applied to the thin rings ($h \ll 2R$) considered in this paper. It may be true, however, that their equations are more accurate where the thin ring approximation does not apply. Wittrick derived equations for helical coils of large pitch which can be reduced to (9) and (10) when the pitch is zero [11].

For very large values of the mean radius R, a ring segment approaches a straight beam. In the limit for $R \to \infty$, (9) and (10) can be easily reduced to the familiar uncoupled classical extensional theory of bars and the Timoshenko beam equations (flexure and shear).

Classical extensional theory of bars:

$$\frac{\partial P}{\partial S} = \rho A \frac{\partial^2 v}{\partial t^2},$$

$$P = EA \frac{\partial v}{\partial S}.$$

Timoshenko beam equations:

$$\frac{\partial Q}{\partial S} = \rho A \frac{\partial^2 u}{\partial t^2},$$

$$\frac{\partial M}{\partial S} - Q = \rho I \frac{\partial^2 \psi}{\partial t^2},$$

$$M = EI \frac{\partial \psi}{\partial S},$$

$$Q = \kappa^2 GA\left(\psi + \frac{\partial u}{\partial S}\right).$$

HARMONIC WAVES UNDER THE THREE-MODE APPROXIMATION

In order to determine the accuracy of the three-mode approximation, we solve the problem of harmonic waves propagating around the ring and compare the result with that of the exact solution. Substitution of (10) into (9) gives the three-mode displacement equations of motion:

$$\begin{aligned} &\kappa^2 GA[\partial\psi/\partial S + \partial^2 u/\partial S^2 - (\partial v/\partial S)/R] - EA(u/R + \partial v/\partial S)/R = \rho A\, \partial^2 u/\partial t^2, \\ &EA[(\partial u/\partial S)/R + \partial^2 v/\partial S^2] + \kappa^2 GA(\psi + \partial u/\partial s - v/R)/R = \rho A\, \partial^2 v/\partial t^2, \\ &EI(\partial^2\psi/\partial S^2) - \kappa^2 GA(\psi + \partial u/\partial S - v/R) = \rho I\, \partial^2\psi/\partial t^2. \end{aligned} \tag{12}$$

Solutions of (12) are given by

$$\begin{aligned} u &= B\exp[i(\omega t - \xi S + \pi/2)], \\ v &= C\exp[i(\omega t - \xi S)], \\ \psi &= (D/R)\exp[i(\omega t - \xi S)], \end{aligned} \tag{13}$$

which describe harmonic waves propagating around the ring. Substitution of (13) into (12) yields three simultaneous equations on the constants B, C and D:

$$\begin{bmatrix} -\alpha z^2 - 1 + \Omega^2 & (1+\alpha)z & -\alpha z \\ (1+\alpha)z & -z^2 - \alpha + \Omega^2 & \alpha + \beta\Omega^2 \\ -\alpha z & \alpha + \beta\Omega^2 & -\beta z^2 - \alpha + \beta\Omega^2 \end{bmatrix} \begin{bmatrix} B \\ C \\ D \end{bmatrix} = 0, \tag{14}$$

where $\alpha = G/E$, $\beta = I/AR^2$, are constants and $\Omega(=\omega R/C_0)$ and $z(=\xi R)$ are the non-dimensional frequency and wave number respectively. The constant $C_0 = (E/\rho)^{1/2}$ is the velocity of extensional waves in straight bars. For non-trivial solutions, the determinant of the symmetric matrix in (14) is set equal to zero which yields the frequency equation

$$\beta^2\Omega^6 - [\alpha + (2+\alpha)\beta^2 z^2]\Omega^4 + [\alpha + \alpha z^2 + \beta^2(2\alpha + 1)z^4]\Omega^2 - \beta^2\alpha(z^2 - 1)^2 z^2 = 0, \tag{15}$$

the solution of which is plotted in Fig. 4 for $\nu = 0{\cdot}25$, $R/h = 10{\cdot}5$. Note the close agreement with the exact solution. For a given frequency, there are three values of z^2 that satisfy (15). Figure 4 shows only half the entire dispersion diagram. Omitted are those curves for which the group velocity is negative. The slope at $\Omega = 0$, $z = 0$ is $h/\sqrt{12}\,R$, and at $\Omega = 0$, $z = \pm 1$, it is $h/\sqrt{6}\,R$. At the cut-off frequencies of the extensional (branch $2R$) and thickness-shear (branch $3R$) branches, the slopes are zero. All of these values are identical with the exact

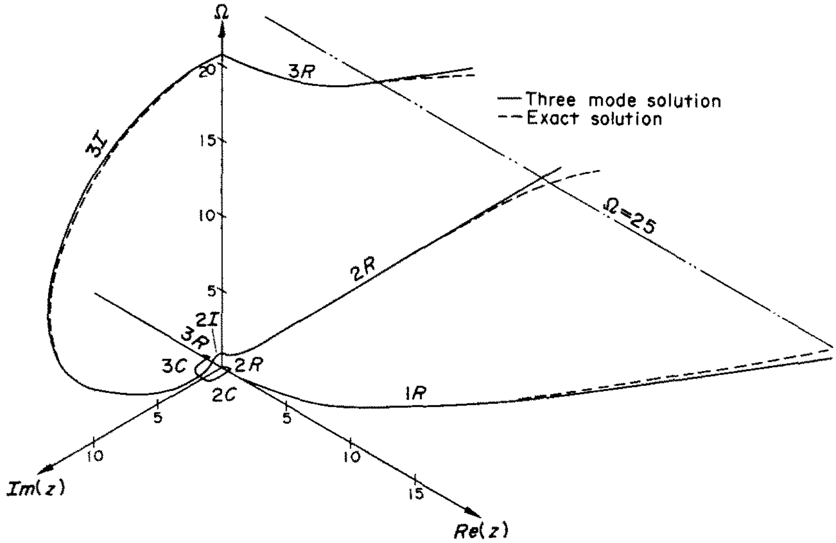

Fig. 4. Dispersion diagrams of elastic, circular ring for three-mode theory and exact solution, $\nu = 0{\cdot}25$, $R/h = 10{\cdot}5$.

results for $h/2R \ll 1$. Amplitude ratios for any point on the dispersion curves may be obtained from (14) and (15). A more detailed discussion of the interesting aspects of Fig. 4 can be found in Ref.[1].

TWO-MODE THEORY

Suppression of shear deformation and rotatory inertia reduces the Timoshenko beam equations to the classical Bernoulli–Euler equations[12]. In the same way with the ring, we set $\rho I\,\partial^2\psi/\partial t^2 = 0$ (rotatory inertia) in (9) and $\psi + \partial u/\partial S - v/R = 0$ (shear strain) in (10) while allowing $\kappa^2 GA \to \infty$, providing for finite, non-zero shear force $Q = \partial M/\partial S$. The result is the set of two-mode equations coupling flexure and extension:

$$\begin{aligned} \frac{\partial^2 M}{\partial S^2} - \frac{P}{R} &= \rho A\,\frac{\partial^2 u}{\partial t^2}, \\ \frac{\partial P}{\partial S} + \frac{1}{R}\frac{M}{S} &= \rho A\,\frac{\partial^2 v}{\partial t^2}, \end{aligned} \tag{16}$$

$$\begin{aligned} P &= EA(u/R + \partial v/\partial S), \\ M &= EI[(1/R)\,\partial v/\partial S - \partial^2 u/\partial S^2]. \end{aligned} \tag{17}$$

The appropriate boundary conditions are still obtained from (11) where P and M are now related to the displacements as given above; also

$$Q = EI\,[(1/R)\partial^2 v/\partial S^2 - \partial^3 u/\partial S^3] \quad \text{and} \quad \psi = v/R - \partial u/\partial S.$$

Waltking[5] seems to have been the first to derive these equations and he was followed by Morley[9]. Earlier, Lamb[13] had the same equations except that instead of (17b) he presented $M = EI(-u/R - \partial^2 u/\partial S^2)$ which in this equation alone assumes zero extension of the mid-line. With the bending moment so defined, a unique solution is no longer insured.

Furthermore, the matrix from which the frequency equation is derived becomes assymetrical; however, this has negligible effect on the calculated frequencies. Philipson [14] and Graff[15] also had this type of moment equation and in addition, Philipson included rotatory inertia which is extraneous without also including shear deformation. Buckens[16] worked out an elaborate description of shear based on static considerations. His matrix is highly assymetrical.

Substitution of (17) into (16) yields the two-mode equations of motion:

$$\begin{aligned} &EI[(1/R)\,\partial^3 v/\partial S^3 - \partial^4 u/\partial S^4] - (EA/R)(u/R + \partial v/\partial S) = \rho A\,\partial^2 u/\partial t^2, \\ &EA[(1/R)\,\partial u/\partial S + \partial^2 v/\partial S^2] + (EI/R)[(1/R)\,\partial^2 v/\partial S^2 - \partial^3 u/\partial S^3] = \rho A\,\partial^2 v/\partial t^2. \end{aligned} \tag{18}$$

The wave solution

$$u = B\exp[i(\omega t - \xi S + \pi/2)],$$
$$v = C\exp[i(\omega t - \xi S)],$$

yields from (18) two simultaneous equations on B and C:

$$\begin{bmatrix} -\beta z^4 - 1 + \Omega^2 & z + \beta z^3 \\ z + \beta z^3 & -z^2(1+\beta) + \Omega^2 \end{bmatrix} \begin{bmatrix} B \\ C \end{bmatrix} = 0,$$

from which the frequency equation

$$\Omega^4 - (1 + z^2 + \beta^2 z^4)\Omega^2 + \beta(z^2 - 1)^2 z^2 = 0, \tag{19}$$

is obtained. The solution of (19) is plotted in Fig. 5 for $R/h = 10{\cdot}5$ with the exact solution shown also. The slopes match at the key points discussed previously, though the frequency range of application is much less than for the three-mode theory due to the neglect of shear deformation and rotatory inertia. The frequency range is the same as for the Bernoulli–Euler beam, i.e. approx 1/15 of the thickness–shear frequency for most materials.

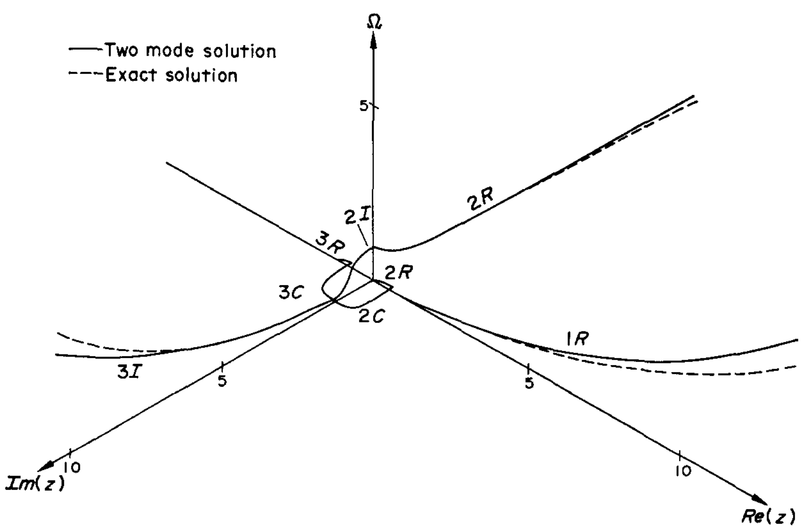

Fig. 5. Dispersion diagrams of elastic, circular ring for two-mode theory and exact solution, $R/h = 10{\cdot}5$.

A ONE-MODE FLEXURE THEORY

We can reduce the two-mode equations to a flexural theory by suppressing the extensional strain so that $u/R + \partial v/\partial S = 0$ in (17a) while $EA \to \infty$, which allows a finite, non-zero tangential force P. Then $M = -EI(u/R^2 + \partial^2 u/\partial S^2)$ and from (16a),

$$P = -REI[(1/R^2)(\partial^2 u/\partial S^2) + \partial^4 u/\partial S^4] - R\rho A\, \partial^2 u/\partial t^2,$$

while $Q = -EI[(1/R^2)\,\partial u/\partial S - \partial^3 u/\partial S^3]$. Substitution of the new expression for P into (16b), differentiated once with respect to S, yields one equation of motion

$$\frac{\partial^4 M}{\partial S^4} + \frac{1}{R^2}\frac{\partial^2 M}{\partial S^2} = \rho A \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 u}{\partial S^2} - \frac{u}{R}\right). \tag{20}$$

The appropriate boundary conditions are again given by (11) where P, M and Q are now defined as above.

The displacement equation of motion is obtained by substituting the M defined above into (20) which gives

$$EI[(1/R^2)\,\partial^4 u/\partial S^4 + \partial^6 u/\partial S^6] + (EI/R^2)[(1/R^2)\,\partial^2 u/\partial S^2 + \partial^4 u/\partial S^4] + \rho A\, \partial^2(\partial^2 u/\partial S^2 - u/R^2)/\partial t^2 = 0. \tag{21}$$

With the wave solution $u = B\exp[i(\omega t - \xi S)]$, we obtain the frequency equation

$$\Omega^2 = \frac{I}{AR^2}\left[\frac{z^2(z^2-1)^2}{z^2+1}\right], \tag{22}$$

which is the equation first derived by Hoppe[6].

Dispersion curves corresponding to solutions of (22) are plotted in Fig. 6. This one mode theory may be used for frequencies below the ring mode frequency ($\Omega = 1$). The match with the exact flexure branch is just as good as the two-mode theory, but the frequency range must be more limited due to the neglect of extensional strain.

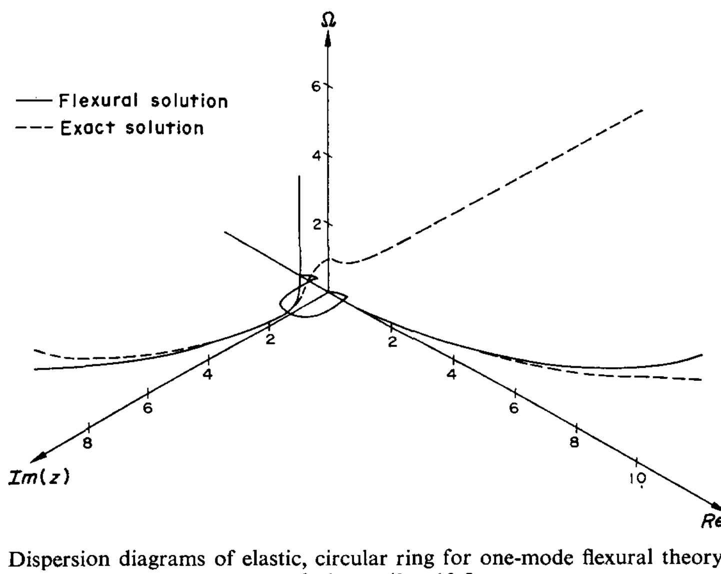

Fig. 6. Dispersion diagrams of elastic, circular ring for one-mode flexural theory and exact solution, $R/h = 10{\cdot}5$.

In addition to Hoppe, (22) has been derived by several authors[17–19], and for coils of large pitch, Love[20] and Michel[21] presented development equivalent to this theory for zero pitch.

A ONE-MODE THEORY OF UNCOUPLED EXTENSION

Uncoupled equations of extensional motion can also be obtained from (16) and (17) by setting $v = -R\,\partial u/\partial S$, while in (17b) $EI \to 0$. Then (17) becomes $M = 0$ and $P = EA(u/R + \partial v/\partial S)$ (unchanged) and (16) become

$$-\frac{P}{R} = \rho A \frac{\partial^2 u}{\partial t^2}, \qquad \frac{\partial P}{\partial S} = \rho A \frac{\partial^2 v}{\partial t^2}. \tag{23}$$

Combination of these equations yields one independent displacement equation of motion, either

$$(EA/R)(-u/R + R\,\partial^2 u/\partial S^2) = \rho A\,\partial^2 u/\partial t^2,$$

or

$$(EA/R)(-v/R + R\,\partial^2 v/\partial S^2) = \rho A\,\partial^2 v/\partial t^2. \tag{24}$$

A unique solution for a bounded ring segment is obtained if either P or v is specified at the two ends. With the solution $v = C\exp[i(\omega t - \xi S)]$, we obtain from (24) the frequency equation

$$\Omega^2 = 1 + z^2, \tag{25}$$

which is plotted in Fig. 7. The frequency range of good match with the exact solution is very

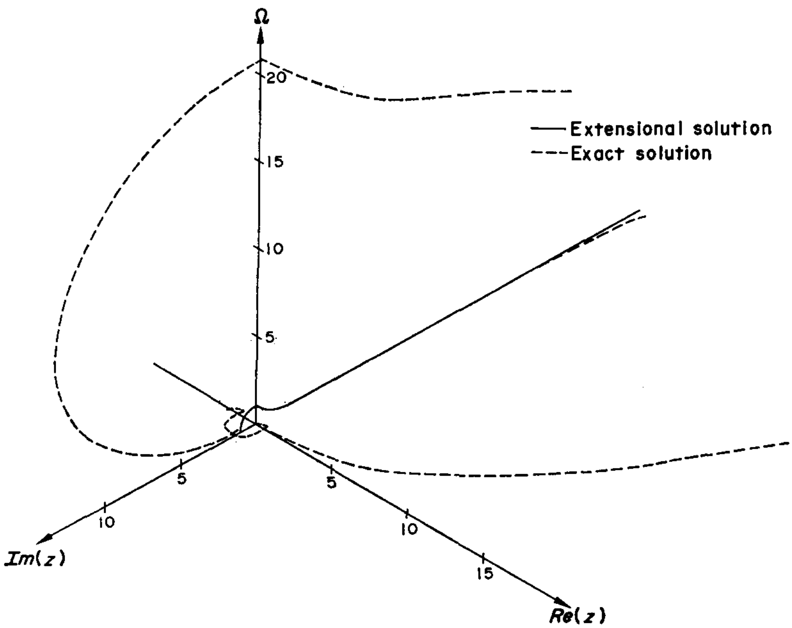

Fig. 7. Dispersion diagrams of elastic, circular ring for one-mode extensional theory and exact solution, $R/h = 10{\cdot}5$.

broad, and is quite useful when the flexural branch can be ignored. Hoppe[6] again seems to be the first to have derived (25). Love[22] presented equations equivalent to (24); he and Lamb[17] also obtained(25). Filipzyński[23] and Lyubavin and Petrov[24] reported experiments confirming (25).

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Абстракт — Из уравнений линейной упругости определяются три уровни приближенных теорий для деформации в плоскости и движния тонкого, круглого кольца. Точность каждой теории определена путем сравнения с решениями гармонических волн в рамках решения теории упругости. Устанавливаются граничные условия для единственности. Результаты можно использовать для винтовой спирали малого шага и цилиндрической оболочки, когда уравнения преобразованы к плоской деформации.